Analytic quantum weak coin flipping protocols with arbitrarily small bias

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Secure two-party computation

Two parties jointly compute an arbitrary function on their inputs without sharing the values of their inputs with the other

Classical

Oblivious Transfer $\Rightarrow$ Bit Commitment $\Rightarrow$ Coin Flipping

Perfect security impossible without extra assumptions (e.g. computational hardness)

Quantum

Oblivious Transfer $\Leftrightarrow$ Bit Commitment $\Rightarrow$ Coin Flipping

Perfect security is impossible (non-relativistic)

Quantum weak coin flipping is the strongest known primitive with arbitrarily perfect security

Coin flipping[1]

over the telephone

Two distrustful parties, Alice and Bob, wish to remotely generate an unbiased random bit.

- **Strong Coin Flipping** (SCF)
  The parties do not know a priori the preferred outcome of the other
- **Weak Coin Flipping** (WCF)
  The parties have a priori known opposite preferred outcomes

[1] M. Blum, SIGACT News 15.1, pp.23-27 (1983).

Protocol features

Honest is a player who follows the protocol exactly as described.

A	**B**	**Feature**	**Pr(A wins)**	**Pr(B wins)**
Honest	Honest	Correctness	$P_A = 1/2$	$P_B = 1/2$
Cheats	Honest	A can bias	P_A^*	$1 - P_A^*$
Honest	Cheats	B can bias	$1 - P_B^*$	P_B^*
Cheats	Cheats	No protocol	–	–

A protocol has **bias** ϵ if neither player can force their desired outcome with probability higher than $\frac{1}{2} + \epsilon$, i.e. the bias is the smallest ϵ such that $P_A^*, P_B^* \leq \frac{1}{2} + \epsilon$.

Bounds and best explicit protocols

Classical

Completely insecure $\epsilon = \frac{1}{2}$, unless extra assumptions are made

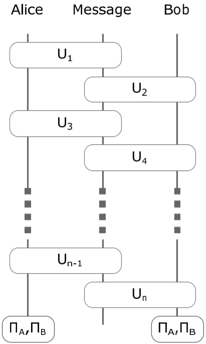
Quantum

	Bound	Protocol
SCF	$\epsilon \geq \frac{1}{\sqrt{2}} - \frac{1}{2}$ [1]	$\epsilon \to \frac{1}{\sqrt{2}} - \frac{1}{2}$ [2] and $\epsilon = \frac{1}{4}$ [3]
WCF	$\epsilon \to 0$ [4,5]	$\epsilon = \frac{1}{10}$ [6], numerically $\epsilon \to 0$ [6]

[1] A. Y. Kitaev, QIP workshop (2003).
[2] A. Chailloux and I. Kerenidis, 50th FOCS, pp. 527-533 (2009).
[3] A. Ambainis, J Comp and Sys Sci 68.2, pp. 398-416 (2004).
[4] C. Mochon, arXiv:0711.4114 (2007).
[5] D. Aharonov, A. Chailloux, M. Ganz, I. Kerenidis and L. Magnin, SIAM J Comp 45.3, pp. 633-679 (2016).

[6] A. S. Arora, J. Roland and S. Weis, 51st ACM SIGACT STOC, pp. 205-216 (2019).

Protocol description



Variables involved: ρ, U

Two SDPs

- P_A^* is an SDP in ρ_B: $P_A^* = \max(\mathrm{tr}(\Pi_A \rho_B))$ s.t. the honest player (Bob) follows the protocol.
- Similarly for P_B^*.

Dual: $\rho \leftrightarrow Z$, $\max \leftrightarrow \min$, $P^* = \max \leftrightarrow P^* \leq$ certificate

A **new framework** is needed permitting us to find **both** the protocol and its bias.

Time-dependent point games* (TDPG)

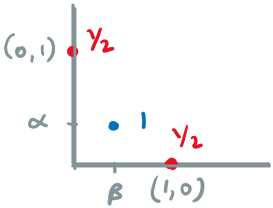
Sequence of frames including points on $x - y$ plane with probability weights assigned

- Starting points: $(0,1)$ and $(1,0)$ with $p = 1/2$.
- Transitions between frames:

$$\sum_z p_z = \sum_{z'} p_{z'}$$

$$\sum_z \frac{\lambda z}{\lambda + z} p_z \leq \sum_{z'} \frac{\lambda z'}{\lambda + z'} p_{z'}, \forall \lambda \geq 0$$

- Final point (β, α) with $p = 1$.

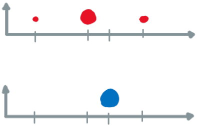


* Mochon in arXiv:0711.4114 attributes the point-game formalism to A. Y. Kitaev.

Examples of allowed moves

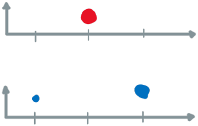
Merge ($n_g \to 1$):

$$\langle x_g \rangle \leq x_h$$



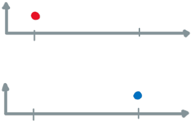
Split ($1 \to n_h$):

$$\frac{1}{x_g} \geq \left\langle \frac{1}{x_h} \right\rangle$$



Raise ($n_g = n_h \to n_h$):

$$x_{g_i} \leq x_{h_i}$$



Transitions expressible by matrices (EBM)

Consider a Hermitian matrix $Z \geq 0$ and let $\Pi^{[z]}$ be the projector on the eigenspace of the eigenvalue z. Then $Z = \sum_z z\Pi^{[z]}$. Let $|\psi\rangle$ be a vector (not necessarily normalised). We define the function $\text{Prob}[Z, |\psi\rangle] : [0, \infty) \to [0, \infty)$ with finite support as

$$\text{Prob}[Z, |\psi\rangle](z) = \begin{cases} \langle\psi|\Pi^{[z]}|\psi\rangle \text{ if } z \in \text{spectrum}(Z) \\ 0 \quad \text{otherwise.} \end{cases}$$

Let $g, h : [0, \infty) \to [0, \infty)$ be two functions with finite supports. The line transition $g \to h$ is called **EBM** if there exist two matrices $0 \leq G \leq H$ and a vector $|\psi\rangle$ such that:

$$g = \text{Prob}[G, |\psi\rangle] \text{ and } h = \text{Prob}[H, |\psi\rangle].$$

For each EBM TDPG there exists a WCF protocol with $P_A^* \leq \alpha, P_B^* \leq \beta$.

Time-independent point games (TIPG)

For an EBM *transition* $g \to h$, we define the EBM *function* $g - h$.

The set of EBM functions is the same (up to closures) as the set of *valid* functions.

A function $f(x)$ is *valid* if $\sum_x f(x) = 0$ and $\sum_x \frac{f(x)}{\lambda + x} \leq 0, \forall \lambda \geq 0$.

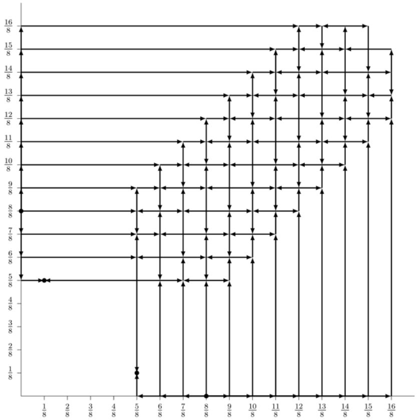
For each TIPG there exists an EBM TDPG with the same final frame

Existence of a WCF protocol with $\epsilon \to 0$[1]

Family of TIPG[2] approaching bias

$$\epsilon = \frac{1}{4k+2},$$

where $2k$ is the number of points involved in the main move of the point game



[1] C. Mochon, arXiv:0711.4114 (2007).

[2] Picture from P. Høyer and E. Pelchat, MA thesis, University of Calgary (2013).

Equivalent frameworks and the proof of existence[1,2]

Protocol

Constructive $\Downarrow$ $\Uparrow$ Non-constructive

Time Dependent Point Game (TDPG)

Constructive $\Downarrow$ $\Uparrow$ Constructive

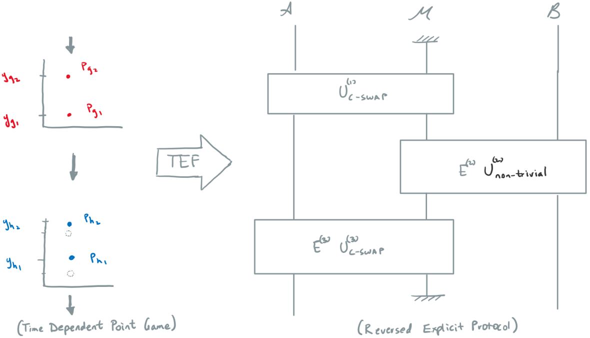
Time Independent Point Game (TIPG)

[1] C. Mochon, arXiv:0711.4114 (2007).

[2] D. Aharonov, A. Chailloux, M. Ganz, I. Kerenidis and L. Magnin, SIAM J Comp 45.3, pp. 633-679 (2016).

TDPG-to-explicit-protocol framework (TEF)[1]

Conversion of a TDPG to an explicit WCF protocol with the corresponding bias, given that for every transition of the TDPG, a unitary satisfying certain constraints can be found



[1] A. S. Arora, J. Roland and S. Weis, 51st ACM SIGACT STOC, pp. 205-216 (2019).

TEF constraints

U is a unitary* matrix acting on $\text{span}\{|g_1\rangle, |g_2\rangle, \ldots, |h_1\rangle, |h_2\rangle, \ldots\}$, s. t.

$$U|v\rangle = |w\rangle \quad \text{and} \quad \sum_{i=1}^{n_h} x_{h_i} |h_i\rangle\langle h_i| - \sum_{i=1}^{n_g} x_{g_i} E_h U |g_i\rangle\langle g_i| U^\dagger E_h \geq 0,$$

with $|v\rangle := \frac{\sum_i \sqrt{p_{g_i}}|g_i\rangle}{\sqrt{\sum_i p_{g_i}}}$ and $|w\rangle := \frac{\sum_i \sqrt{p_{h_i}}|h_i\rangle}{\sqrt{\sum_i p_{h_i}}}$, $\{\{|g_i\rangle\}_{i=1}^{n_g}, \{|h_i\rangle\}_{i=1}^{n_h}\}$ orthonormal and $E_h := \sum_{i=1}^{n} |h_i\rangle\langle h_i|$. Also, x_{g_i} and x_{h_i} are the coordinates of the n_g and n_h points of the initial and final frame, respectively, with corresponding probability weights p_{g_i} and p_{h_i}

Using TEF[1] a protocol with $\epsilon = \frac{1}{10}$ was constructed analytically and an algorithm was proposed to numerically construct U for lower bias

* it is sufficient to consider orthogonal matrices

[1] A. S. Arora, J. Roland and S. Weis, 51st ACM SIGACT STOC, pp. 205-216 (2019).

$f-$ assignment[1]

Given a set of real coordinates $0 \leq x_1 < x_2 \cdots < x_n$ and a polynomial of degree at most $n-2$ satisfying $f(-\lambda) \geq 0$ for all $\lambda \geq 0$, an *f-assignment* is given by the function

$$t = \sum_{i=1}^{n} \underbrace{\frac{-f(x_i)}{\prod_{j \neq i}(x_j - x_i)}}_{=:p_i} [x_i] = h - g,$$

where h contains the positive part of t and g the negative part (without any common support), viz. $h = \sum_{i:p_i>0} p_i \, [x_i]$ and $g = \sum_{i:p_i<0} (-p_i) \, [x_i]$.

- An assignment is *balanced* if the number of points with negative weights, $p_i < 0$, equals the number of points with positive weights, $p_i > 0$. An assignment is *unbalanced* if it is not balanced.
- When f is a monomial, viz. has the form $f(x) = cx^q$, where $c > 0$ and $q \geq 0$, we call the assignment a *monomial assignment*.
- A monomial assignment is *aligned* if the degree of the monomial is an even number ($q = 2(b-1)$, $b \in \mathbb{N}$). A monomial assignment is *misaligned* if it is not aligned.

[1] C. Mochon, arXiv:0711.4114 (2007).

The $f-$assignment as a sum of monomial assignments

Consider a set of real coordinates satisfying $0 \leq x_1 < x_2 \cdots < x_n$ and let $f(x) = (r_1 - x)(r_2 - x) \ldots (r_k - x)$ where $k \leq n - 2$. Let $t = \sum_{i=1}^{n} p_i \, [x_i]$ be the corresponding f-assignment.

Then

$$t = \sum_{l=0}^{k} \alpha_l \left(\sum_{i=1}^{n} \frac{-(-x_i)^l}{\prod_{j \neq i}(x_j - x_i)} \, [x_i] \right),$$

where $\alpha_l \geq 0$.

More precisely, α_l is the coefficient of $(-x)^l$ in $f(x)$.

Solving an assignment

Given an $f-$ assignment $t = \sum_{i=1}^{n_h} p_{h_i} [x_{h_i}] - \sum_{i=1}^{n_g} p_{g_i} [x_{g_i}]$ and an orthonormal basis $\left\{ |g_1\rangle, |g_2\rangle \ldots \left| g_{n_g} \right\rangle, |h_1\rangle, |h_2\rangle \ldots |h_{n_h}\rangle \right\}$, we say that the orthogonal matrix O *solves* t if

$$O |v\rangle = |w\rangle \text{ and } X_h \geq E_h O X_g O^T E_h,$$

where $|v\rangle = \sum_{i=1}^{n_g} \sqrt{p_{g_i}} |g_i\rangle$, $|w\rangle = \sum_{i=1}^{n_h} \sqrt{p_{h_i}} |h_i\rangle$, $X_h = \sum_{i=1}^{n_h} x_{h_i} |h_i\rangle \langle h_i|$, $X_g = \sum_{i=1}^{n_g} x_{g_i} |g_i\rangle \langle g_i|$ and $E_h = \sum_{i=1}^{n_h} |h_i\rangle \langle h_i|$.

Moreover, we say that t has an *effective solution* if $t = \sum_{i \in I} t'_i$ and t'_i has a solution for all $i \in I$, where I is a finite set.

4 types of monomial assignments: balanced/unbalanced – aligned/misaligned

Analytic solution

Balanced and aligned monomial assignments

Let $m = 2b \in \mathbb{Z}, t = \sum_{i=1}^{n} x_{h_i}^m p_{h_i} \left[x_{h_i}\right] - \sum_{i=1}^{n} x_{g_i}^m p_{g_i} \left[x_{g_i}\right]$ a monomial assignment over $0 < x_1 < x_2 \cdots < x_{2n}$, $\{\left|h_1\right\rangle, \left|h_2\right\rangle \ldots \left|h_n\right\rangle, \left|g_1\right\rangle, \left|g_2\right\rangle \ldots \left|g_n\right\rangle\}$ an orthonormal basis, and

$$X_g := \sum_{i=1}^{n} x_{g_i} \left|g_i\right\rangle \left\langle g_i\right| \doteq \operatorname{diag}(\underbrace{0, 0, \ldots 0}_{n \text{ zeros}}, x_{g_1}, x_{g_2} \ldots x_{g_n}),$$

$$X_h := \sum_{i=1}^{n} x_{h_i} \left|h_i\right\rangle \left\langle h_i\right| \doteq \operatorname{diag}(x_{h_1}, x_{h_2} \ldots x_{h_n}, \underbrace{0, 0 \ldots 0}_{n \text{ zeros}}),$$

$$\left|v\right\rangle := \sum_{i=1}^{n} \sqrt{p_{g_i}} \left|g_i\right\rangle \doteq (\underbrace{0, 0, \ldots 0}_{n \text{ zeros}}, \sqrt{p_{g_1}}, \sqrt{p_{g_2}} \ldots \sqrt{p_{g_n}})^T \quad \text{and} \quad \left|v'\right\rangle := (X_g)^b \left|v\right\rangle.$$

$$\left|w\right\rangle := \sum_{i=1}^{n} \sqrt{p_{h_i}} \left|h_i\right\rangle \doteq (\sqrt{p_{h_1}}, \sqrt{p_{h_2}} \ldots \sqrt{p_{h_n}}, \underbrace{0, 0, \ldots 0}_{n \text{ zeros}})^T \quad \text{and} \quad \left|w'\right\rangle := (X_h)^b \left|w\right\rangle,$$

Analytic solution

Balanced and aligned monomial assignments

Then,

$$O := \sum_{i=-b}^{n-b-1} \left(\frac{\Pi^{\perp}_{h_i} (X_h)^i \left|w'\right\rangle \left\langle v'\right| (X_g)^i \Pi^{\perp}_{g_i}}{\sqrt{c_{h_i} c_{g_i}}} + \text{h.c.} \right)$$

satisfies

$$X_h \geq E_h O X_g O^T E_h \quad \textit{and} \quad E_h O \left|v'\right\rangle = \left|w'\right\rangle,$$

where $E_h := \sum_{i=1}^{n} \left|h_i\right\rangle \left\langle h_i\right|$, $c_{h_i} := \left\langle w'\right| (X_h)^i \Pi^{\perp}_{h_i} (X_h)^i \left|w'\right\rangle$, and

$$\Pi^{\perp}_{h_i} := \begin{cases} \text{projector orthogonal to span}\{(X_h)^{-|i|+1} \left|w'\right\rangle, (X_h)^{-|i|+2} \left|w'\right\rangle \ldots, \left|w'\right\rangle\} & i < 0 \\ \text{projector orthogonal to span}\{(X_h)^{-b} \left|w'\right\rangle, (X_h)^{-b+1} \left|w'\right\rangle, \ldots (X_h)^{i-1} \left|w'\right\rangle\} & i > 0 \\ \mathbb{I} & i = 0. \end{cases}$$

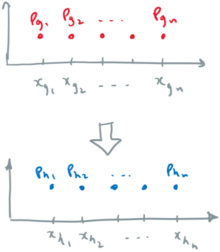
Analogous are the forms of $\Pi^{\perp}_{g_i}$ and c_{g_i}.

The expressions for the solution O for the other possible types of monomial assignments are similar

Analytic solution

Balanced and aligned monomial assignments

Suppose $b=0$.



$$O \quad \text{s.t.} \quad \begin{array}{rcl} |v\rangle & \longmapsto & |w\rangle \\ \Pi_{g_1} X_g |v\rangle & \longmapsto & \Pi_{h_1} X_h |w\rangle \\ \Pi_{g_2} X_g^2 |v\rangle & \longmapsto & \Pi_{h_2} X_h^2 |w\rangle \\ & \vdots & \\ \Pi_{g_n} X_g^n |v\rangle & \longmapsto & \Pi_{h_n} X_h^n |w\rangle \end{array} \quad + \text{ h.c.}$$

Summary and conclusions

- Analytical construction of WCF protocols with arbitrarily close to zero bias
- Our approach is simpler as it avoids the – quite technical – reduction of the problem from EBM to valid functions
- Analytical solutions in fewer dimensions?

Open questions

- Protocols for the Pelchat-Høyer family[1] of point games?
- Given the recent bound on the rounds of communication[2], can we find protocols matching the bounds on resources?
- Noise robustness of the protocols.
- Device independent protocols[3]

[1] P. Høyer and E. Pelchat, MA thesis, University of Calgary (2013).
[2] C. A. Miller, 52nd ACM SIGACT STOC, pp. 916-929 (2020).

[3] N. Aharon, A. Chailloux, I. Kerenidis, S. Massar, S. Pironio and J. Silman, 6th TQC (2011).

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